Encapsulation theory: the minimised, uniformly hidden radial branch.

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Abstract

This paper investigates how the violational elements of a single, radial branch uniformly distributed in hidden elements should be distributed so as to minimise the branch's potential coupling.

Keywords

Encapsulation theory, radial encapsulation, branch, potential coupling.

1. Introduction

As was shown in [3], the potential coupling of a radial branch is not minimised when all its hidden and violational elements are uniformly distributed over all its disjoint primary sets and instead if the violational elements are uniformly distributed then there exists a function which dictates how to non-uniformly distribute the hidden elements such that the potential coupling falls below that of a uniformly-distributed branch.

Similarly, there exists a function which dictates how to distribute the violation elements of a branch so as to minimise its potential coupling given a uniform distribution of hidden elements.

This paper investigates this second, violational distribution function.

This paper considers sets of radial information-hiding only.

2. Selected branch potential couplings

Consider the branch shown in figure 1, which was also investigated in [3], showing a branch of three disjoint primary sets, where the information-hiding and the information-hiding violation of the sets are shown within their representative symbols.

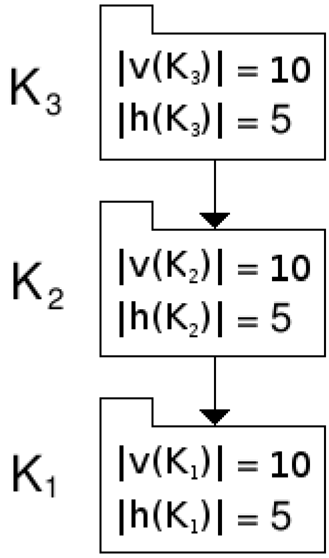

Figure 1: A branch of three disjoint primary sets.

Thus, taking the root set, K_1, for example, we see that K_1 has 10 violational elements, $\left|v(K_1)\right| = 10$, and 5 hidden elements, $\left|h(K_1)\right| = 5$.

As was shown in [3], the branch's potential coupling is: *510+360+210=1080*.

We know that, in the absolute encapsulation context, potential coupling is minimised (ignoring A.M.C.s) when both the violational and hidden element are distributed uniformly over all sets. Let us test to see whether this also holds in the radial encapsulation context: let us move a violational element from K_1 to K_3, see figure 2.

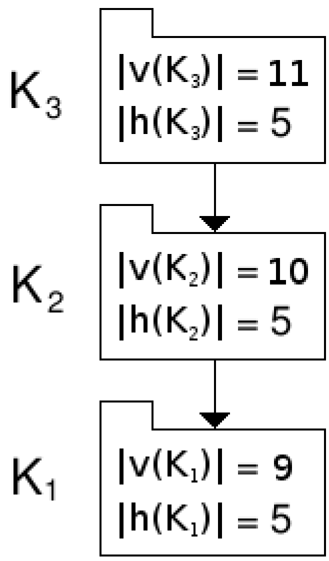

Figure 2: A branch of three disjoint primary sets, one element migrated.

Let us calculate the potential coupling of the branch in figure 2.

The potential coupling of K_3 is the sum of its internal and external potential couplings. The internal potential coupling of K_3 is the total number of contained elements multiplied by this number minus one, i.e., *16x15=240*. The external potential coupling of K_3 is the total number of contained elements multiplied by the total number of violational elements it can see "below" it, i.e., *16x(10+9)=304*. So the potential coupling of K_3 is *240+304=544*.

The internal potential coupling of K_2 is *15x14=210*. K_2 can only see *9* violational elements "below" it, so its external potential coupling is *15x9=135*. So the potential coupling of K_2 is *210+135=345*.

The internal potential coupling of K_1 is *14x13=182* and K_1 has not external potential coupling so its potential coupling is just *182*.

The branch's total potential coupling has now fallen from *1080* to *544+345+182=1071*.

Thus by moving a violational element, thereby rendering the branch non-uniformly distributed in violational elements, we have reduced the branch's potential coupling. We note that the branch is still uniformly distributed in hidden elements. This then raises the obvious question: given a branch that is uniformly distributed in hidden elements, what distribution of violational elements will minimise its potential coupling?

It can be shown that, to attain the minimum potential coupling in this case, the number of violational elements in the i^{th} disjoint primary set is given by the following equation (see proposition 10.18):

$$|v(K_i)| = \frac{n_B}{r_B} + \frac{(2i-3-r_B)|h(K)|}{2}$$

With this equation we can calculate the number of violational elements in each of the three sets examined above to minimise the potential coupling.

K_3 should contain the following number of violational elements:

$$|v(K_3)| = \frac{45}{3} + \frac{(6-3-3)5}{2} = 15$$

K_2 should contain the following number of violational elements:

$$|v(K_2)| = \frac{45}{3} + \frac{(4-3-3)5}{2} = 10$$

K_1 should contain the following number of violational elements:

$$|v(K_2)| = \frac{45}{3} + \frac{(2-3-3)5}{2} = 5$$

The branch thus configured is shown in figure 3.

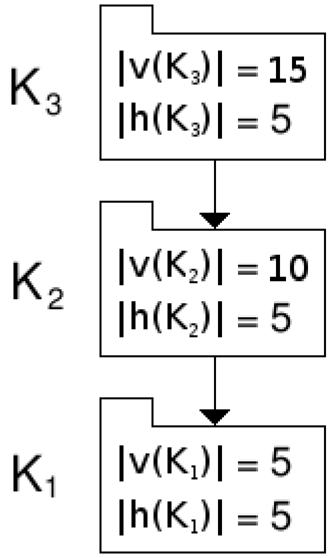

Figure 3: A branch non-uniformly distributed in violational elements with minimised potential coupling.

Calculating the potential coupling of this minimised branch we see the following.

The potential coupling of K_3 is *20x19+20x15=680*.

The potential coupling of K_2 is *15x14+15x5=285*.

The potential coupling of K_1 is *10x9=90.*

The branch's total potential coupling is *680+285+90=1055*, down from the original, uniformly distributed figure of *1080*.

4. Conclusion

When a branch is uniformly distributed in hidden elements, its minimum potential coupling is not achieved by also uniformly distributing its violational elements. Instead, an equation describes the violational element distribution to achieve this minimum potential coupling and this equation suggests placing more violational elements towards the higher part of the branch than the lower.

5. Appendix A

5.2 Definitions

[D10.1] Given a branch *B* of r_B disjoint primary sets, the number of elements in *B*, denoted n_B , is the sum of the number of hidden and violational elements in each disjoint primary set constituting the branch, or:

$$n_B = \sum_{i=1}^{r_B} |K_i|$$

[D10.2] Given a branch *B* of r_B disjoint primary sets, the number of violational elements in *B*, denoted $|V_B|$, is the sum of the number of violational elements in each disjoint primary set constituting the branch, or:

$$|V_B| = \sum_{i=1}^{r_B} |v(K_i)|$$

5.2 Propositions

Proposition 10.1

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the number of elements in the i^{th} primary set, $|K_i|$, is given by:

$$|K_i| = |v(K_i)| + |h(K)|$$

Proof:

By definitions [D1.2], [D1.6] and [D1.7] of [1], the elements in K_i are those in the intersection of K_i with *H* and *V*, or:

$$K_i = v(K_i) \cup h(K_i) \quad \text{(i)}$$

By definition [D1.1] in [1], *H* and *V* are disjoint so taking the cardinality of (i) gives:

$$|K_i| = |v(K_i)| + |h(K_i)| \quad \text{(ii)}$$

We presume that each disjoint primary set has an information-hiding of $|h(K)|$, that is:

$$|h(K)| = |h(K_i)| \forall i \quad \text{(iii)}$$

Substituting (iii) into (ii) gives:

$$|K_i| = |v(K_i)| + |h(K)|$$

QED

Proposition 10.2

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the internal potential coupling of the i^{th} primary set, $|s_{\text{in}}(Q_i)|$, is given by:

$$|s_{\text{in}}(Q_i)| = |h(K)|^2 + |v(K_i)|^2 + 2|h(K)||v(K_i)| - |h(K)| - |v(K_i)|$$

Proof:

By definition [D1.2] in [1], given a primary set Q_i the internal potential coupling $|s_{in}(Q_i)|$ is given by:

$$|s_{\text{in}}(Q_i)| = |K_i|(|K_i| - 1) \quad \text{(i)}$$

By proposition 10.1, the number of elements in the i^{th} primary set, $|K_i|$, is given by:

$$|K_i| = |v(K_i)| + |h(K)| \quad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$|s_{\text{in}}(Q_i)| = (|v(K_i)| + |h(K)|)(|v(K_i)| + |h(K)| - 1)$$

$$= \ |h(K)|^2 + |v(K_i)|^2 + 2|h(K)||v(K_i)| - |h(K)| - |v(K_i)|$$

QED

Proposition 10.3

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the following holds:

$$\sum_{j=1}^{i-1} |v(K_j)| = n_B - |h(K)| r_B - \sum_{j=i}^{r_B} |v(K_j)|$$

Proof:

From definition [D10.1], the number of elements in *B*, denoted n_B, is the sum of hidden and violational elements in each disjoint primary set constituting the branch, or:

$$n_B = \sum_{j=1}^{r_B} |K_j| \qquad \text{(i)}$$

By proposition 10.1, the number of elements in the j^{th} primary set, $|K_j|$, is given by:

$$|K_j| = |v(K_j)| + |h(K)| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$n_B = \sum_{j=1}^{r_B} (|v(K_j)| + |h(K)|)$$

$$= \ \sum_{j=1}^{r_B} |v(K_j)| + \sum_{j=1}^{r_B} |h(K)|$$

$$= \ \sum_{j=1}^{r_B} |v(K_j)| + |h(K)| r_B \qquad \text{(iii)}$$

Taking an arbitrary *i* such that $1 \leq i \leq r_B$, then it follows that:

$$\sum_{j=1}^{r_B} |v(K_j)| = \sum_{j=1}^{i-1} |v(K_j)| + \sum_{j=i}^{r_B} |v(K_j)| \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives:

$$n_B = \sum_{j=1}^{i-1} |v(K_j)| + \sum_{j=i}^{r_B} |v(K_j)| + |h(K)| r_B$$

$$\sum_{j=1}^{i-1} |v(K_j)| = n_B - |h(K)| r_B - \sum_{j=i}^{r_B} |v(K_j)|$$

QED

Proposition 10.4

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the external potential coupling of the i^{th} primary set, $|s_{ex}(Q_i)|$, is given by:

$$|s_{ex}(Q_i)|=n_B|v(K_i)|-|h(K)|r_B|v(K_i)|-|v(K_i)|\sum_{j=i}^{r_B}|v(K_j)|+|h(K)|n_B-|h(K)|^2 r_B-|h(K)|\sum_{j=i}^{r_B}|v(K_j)|$$

Proof:

Consider the root primary set Q_1.

As Q_1 is the root primary set, however, all other disjoint primary sets in this branch are subsets of Q_1, thus the external potential coupling of the root disjoint primary set is 0.

The external potential coupling of the Q_2 is then the number of ordered pairs that may be formed towards K_1.

The external potential coupling of the Q_3 is the number of ordered pairs that may be formed towards K_1 and K_2, etc.

Thus:

$$|s_{ex}(Q_1)|=0$$
$$|s_{ex}(Q_2)|=|K_2||v(K_1)|$$
$$|s_{ex}(Q_3)|=|K_3|(|v(K_2)|+|v(K_1)|)$$
$$|s_{ex}(Q_4)|=|K_4|(|v(K_3)|+|v(K_2)|+|v(K_1)|)$$

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$$|s_{ex}(Q_i)|=|K_i|(|v(K_{i-1})|+|v(K_{i-2})|+...+|v(K_2)|+|v(K_1)|)$$

$$= |K_i|\sum_{j=1}^{i-1}|v(K_j)| \qquad \text{(i)}$$

By proposition 10.3, the following holds:

$$\sum_{j=1}^{i-1}|v(K_j)|=n_B-|h(K)|r_B-\sum_{j=i}^{r_B}|v(K_j)| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$|s_{ex}(Q_i)|=|K_i|(n_B-|h(K)|r_B-\sum_{j=i}^{r_B}|v(K_j)|) \qquad \text{(iii)}$$

By proposition 10.1, the number of elements in the i^{th} primary set, $|K_i|$, is given by:

$$|K_i|=|v(K_i)|+|h(K)| \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives;

$$|s_{ex}(Q_i)|=(|v(K_i)|+|h(K)|)(n_B-|h(K)|r_B-\sum_{j=i}^{r_B}|v(K_j)|)$$

$$= n_B|v(K_i)| - |h(K)|r_B|v(K_i)| - |v(K_i)|\sum_{j=i}^{r_B}|v(K_j)| + |h(K)|n_B - |h(K)|^2 r_B - |h(K)|\sum_{j=i}^{r_B}|v(K_j)|$$

QED

Proposition 10.5

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the external potential coupling of the *i*^th^ primary set, $|s_{ex}(Q_i)|$, is given by:

$$|s_{ex}(Q_i)| = n_B|v(K_i)| - |h(K)|r_B|v(K_i)| - |v(K_i)|^2 - |v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+ |h(K)|n_B - |h(K)|^2 r_B - |h(K)||v(K_i)| - |h(K)|\sum_{j=i+1}^{r_B}|v(K_j)|$$

Proof:

By proposition 10.4, the external potential coupling of the i^{th} primary set, $|s_{ex}(Q_i)|$, is given by:

$$|s_{ex}(Q_i)| = n_B|v(K_i)| - |h(K)|r_B|v(K_i)| - |v(K_i)|\sum_{j=i}^{r_B}|v(K_j)| + |h(K)|n_B - |h(K)|^2 r_B - |h(K)|\sum_{j=i}^{r_B}|v(K_j)|$$

(i)

By definition:

$$\sum_{j=i}^{r_B}|v(K_j)| = |v(K_i)| + \sum_{j=i+1}^{r_B}|v(K_j)| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$|s_{ex}(Q_i)| = n_B|v(K_i)| - |h(K)|r_B|v(K_i)| - |v(K_i)|^2 - |v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+ |h(K)|n_B - |h(K)|^2 r_B - |h(K)||v(K_i)| - |h(K)|\sum_{j=i+1}^{r_B}|v(K_j)|$$

QED

Proposition 10.6

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the potential coupling of the i^{th} primary set, $|s(Q_i)|$, is given by:

$$|s(Q_i)|=|h(K)|^2+|h(K)||v(K_i)|-|h(K)|-|v(K_i)|+n_B|v(K_i)|-|h(K)|r_B|v(K_i)|-|v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+|h(K)|n_B-|h(K)|^2 r_B-|h(K)|\sum_{j=i+1}^{r_B}|v(K_j)|$$

Proof:

By proposition 10.2, the internal potential coupling of the i^{th} primary set, $|s_{\text{in}}(Q_i)|$, is given by:

$$|s_{\text{in}}(Q_i)|=|h(K)|^2+|v(K_i)|^2+2|h(K)||v(K_i)|-|h(K)|-|v(K_i)| \qquad \text{(i)}$$

By proposition 10.5, the external potential coupling of the i^{th} primary set, $|s_{ex}(Q_i)|$, is given by:

$$|s_{ex}(Q_i)|=n_B|v(K_i)|-|h(K)|r_B|v(K_i)|-|v(K_i)|^2-|v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+|h(K)|n_B-|h(K)|^2 r_B-|h(K)||v(K_i)|-|h(K)|\sum_{j=i+1}^{r_B}|v(K_j)| \qquad \text{(ii)}$$

By proposition 8.7 in [3], the potential coupling of Q_i is the sum of the external potential coupling sets of Q_i and the internal potential coupling of Q_i, or:

$$|s(Q_i)|=|s_{ex}(Q_i)|+|s_{in}(Q_i)| \qquad \text{(iii)}$$

Substituting (i) and (ii) into (iii) gives:

$$|s(Q_i)|=|h(K)|^2+|v(K_i)|^2+2|h(K)||v(K_i)|-|h(K)|-|v(K_i)|$$
$$+n_B|v(K_i)|-|h(K)|r_B|v(K_i)|-|v(K_i)|^2-|v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+|h(K)|n_B-|h(K)|^2 r_B-|h(K)||v(K_i)|-|h(K)|\sum_{j=i+1}^{r_B}|v(K_j)|$$

$$= |h(K)|^2+|h(K)||v(K_i)|-|h(K)|-|v(K_i)|+n_B|v(K_i)|-|h(K)|r_B|v(K_i)|-|v(K_i)|\sum_{j=i+1}^{r_B}|v(K_j)|$$
$$+|h(K)|n_B-|h(K)|^2 r_B-|h(K)|\sum_{j=i+1}^{r_B}|v(K_j)|$$

QED

Proposition 10.7

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the number of violational elements in the first disjoint primary set, $|v(K_1)|$, is given by:

$$|v(K_1)|=n_B-\sum_{j=2}^{r_B}|v(K_j)|-|h(K)|r_B$$

Proof:

From definition [D10.1], the number of elements in *B*, denoted n_B, is the sum of hidden and violational elements in each disjoint primary set constituting the branch, or:

$$n_B=\sum_{j=1}^{r_B}|K_j| \qquad \text{(i)}$$

By proposition 10.1, the number of elements in the j^{th} primary set, $|K_j|$, is given by:

$$|K_j|=|v(K_j)|+|h(K)| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$n_B=\sum_{j=1}^{r_B}(|v(K_j)|+|h(K)|)$$

$$=\sum_{j=1}^{r_B}|v(K_j)|+\sum_{j=1}^{r_B}|h(K)|$$

$$=\sum_{j=1}^{r_B}|v(K_j)|+|h(K)|r_B \qquad \text{(iii)}$$

By definition:

$$\sum_{j=1}^{r_B} |v(K_j)| = |v(K_1)| + \sum_{j=2}^{r_B} |v(K_j)| \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives:

$$n_B = |v(K_1)| + \sum_{j=2}^{r_B} |v(K_j)| + |h(K)| r_B$$

$$|v(K_1)| = n_B - \sum_{j=2}^{r_B} |v(K_j)| - |h(K)| r_B$$

QED

Proposition 10.8

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the square of the number of violational elements in the first disjoint primary set, $|v(K_1)|^2$, is given by:

$$|v(K_1)|^2 = \sum_{j=2}^{r_B} |v(K_j)| \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 r_B^2 - 2|h(K)| n_B r_B$$
$$-2 n_B \sum_{j=2}^{r_B} |v(K_j)| + 2|h(K)| r_B \sum_{j=2}^{r_B} |v(K_j)|$$

Proof:

By proposition 10.7, the number of violational elements in the first disjoint primary set, $|v(K_1)|$, is given by:

$$|v(K_1)| = n_B - \sum_{j=2}^{r_B} |v(K_j)| - |h(K)| r_B \qquad \text{(i)}$$

Squaring (i) gives:

$$|v(K_1)|^2 = (n_B - \sum_{j=2}^{r_B} |v(K_j)| - |h(K)| r_B)^2$$

$$= \begin{array}{c} n_B{}^2 + \sum_{j=2}^{r_B} |v(K_j)| \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 r_B^2 - 2|h(K)| n_B r_B - 2 n_B \sum_{j=2}^{r_B} |v(K_j)| \\ +2|h(K)| r_B \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 + 2|h(K)| n_B - 2|h(K)|^2 r_B \\ -|h(K)| - n_B + r_B |h(K)| + \sum_{j=2}^{r_B} |v(K_j)| \end{array}$$

QED

Proposition 10.9

Given a branch B of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the potential coupling of the first primary set, $|s(Q_1)|$, is given by:

$$\begin{aligned}|s(Q_1)| = {} & {n_B}^2 + \sum_{j=2}^{r_B} |v(K_j)| \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 r_B^2 - 2|h(K)| n_B r_B - 2 n_B \sum_{j=2}^{r_B} |v(K_j)| \\ & + 2|h(K)| r_B \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 + 2|h(K)| n_B - 2|h(K)|^2 r_B \\ & - 2|h(K)| \sum_{j=2}^{r_B} |v(K_j)| - |h(K)| - n_B + r_B |h(K)| + \sum_{j=2}^{r_B} |v(K_j)|\end{aligned}$$

Proof:

By definition [D1.2] in [1], given primary set Q_1 the internal potential coupling $|s_{\text{in}}(Q_1)|$ is given by:

$$|s_{\text{in}}(Q_1)| = |K_1|(|K_1| - 1) \qquad \text{(i)}$$

By proposition 10.1, the number of elements in the first primary set, $|K_1|$, is given by:

$$|K_1| = |v(K_1)| + |h(K)| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$\begin{aligned}|s_{\text{in}}(Q_1)| &= (|v(K_1)| + |h(K)|)(|v(K_1)| + |h(K)| - 1) \\ &= |h(K)|^2 + |v(K_1)|^2 + 2|h(K)||v(K_1)| - |h(K)| - |v(K_1)| \qquad \text{(iii)}\end{aligned}$$

By proposition 10.7, the number of violational elements in the first disjoint primary set, $|v(K_1)|$, is given by:

$$|v(K_1)| = n_B - \sum_{j=2}^{r_B} |v(K_j)| - |h(K)| r_B \qquad \text{(iv)}$$

By proposition 10.8, the square of the number of violational elements in the first disjoint primary set, $|v(K_1)|^2$, is given by:

$$\begin{aligned}|v(K_1)|^2 = {} & \sum_{j=2}^{r_B} |v(K_j)| \sum_{j=2}^{r_B} |v(K_j)| + |h(K)|^2 r_B^2 - 2|h(K)| n_B r_B \\ & - 2 n_B \sum_{j=2}^{r_B} |v(K_j)| + 2|h(K)| r_B \sum_{j=2}^{r_B} |v(K_j)|\end{aligned} \qquad \text{(v)}$$

Substituting (iv) and (v) into (iii) gives:

$$\left|s_{\text{in}}(Q_1)\right| = {n_B}^2 + \sum_{j=2}^{r_B} \left|v(K_j)\right| \sum_{j=2}^{r_B} \left|v(K_j)\right| + \left|h(K)\right|^2 r_B^2 - 2\left|h(K)\right| n_B r_B - 2 n_B \sum_{j=2}^{r_B} \left|v(K_j)\right|$$
$$+ 2\left|h(K)\right| r_B \sum_{j=2}^{r_B} \left|v(K_j)\right| + \left|h(K)\right|^2 + 2\left|h(K)\right| n_B - 2\left|h(K)\right|^2 r_B \qquad \text{(vi)}$$
$$- 2\left|h(K)\right| \sum_{j=2}^{r_B} \left|v(K_j)\right| - \left|h(K)\right| - n_B + r_B \left|h(K)\right| + \sum_{j=2}^{r_B} \left|v(K_j)\right|$$

By definition, the first disjoint primary set has no external potential coupling, therefore $\left|s(Q_1)\right| = \left|s_{\text{in}}(Q_1)\right|$ and therefore (vi) is also the equation for the potential coupling of the first disjoint primary set.

QED

Proposition 10.10

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $\left|h(K)\right|$, the potential coupling of *B* is minimised when:

$$\sum_{i=1}^{r_B} \frac{\partial\left|s(Q_1)\right|}{\partial\left|v(K_i)\right|} + \sum_{k=2}^{r_B} \sum_{i=1}^{r_B} \frac{\partial\left|s(Q_k)\right|}{\partial\left|v(K_i)\right|} = 0$$

Proof:

To find the number of violational elements that minimises the potential coupling of branch *B* we must differentiate the potential coupling of *B* with respect to the total number of violational elements and set equal to zero, that is:

$$\frac{d\left|s(B)\right|}{d\left|V_B\right|} = 0 \qquad \text{(i)}$$

By definition [D10.2], the number of violational elements in *B*, denoted $\left|V_B\right|$, is the sum of the number of violational elements in each disjoint primary set constituting the branch, or:

$$\left|V_B\right| = \sum_{i=1}^{r_B} \left|v(K_i)\right| \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$\sum_{i=1}^{r_B} \frac{\partial\left|s(B)\right|}{\partial\left|v(K_i)\right|} = 0 \qquad \text{(iii)}$$

By definition I SHOULD PROVE THIS ???????????????????????xxx, the potential coupling of branch B is equal to the sum of the potential coupling of all its primary sets, or:

$$\left|s(B)\right| = \sum_{m=1}^{r_B} \left|s(Q_m)\right| \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives:

$$\sum_{m=1}^{r_B}\sum_{i=1}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}=0 \qquad \text{(v)}$$

$$= \sum_{i=1}^{r_B}\frac{\partial|s(Q_1)|}{\partial|v(K_i)|}+\sum_{m=2}^{r_B}\sum_{i=1}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}=0$$

QED

Proposition 10.11

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the sum of the potential coupling of the first primary set Q_1 with respect to the number of violational elements in all disjoint primary sets of *B* is given by:

$$\sum_{i=1}^{r_B}\frac{\partial|s(Q_1)|}{\partial|v(K_i)|}=-2r_B|v(K_1)|+2|v(K_1)|-2|h(K)|r_B+2|h(K_i)|+r_B-1$$

Proof:

By proposition 10.9, the potential coupling of the first primary set, $|s(Q_1)|$, is given by:

$$|s(Q_1)|=n_B{}^2+\sum_{j=2}^{r_B}|v(K_j)|\sum_{j=2}^{r_B}|v(K_j)|+|h(K)|^2r_B^2-2|h(K)|n_Br_B-2n_B\sum_{j=2}^{r_B}|v(K_j)|$$

$$+2|h(K)|r_B\sum_{j=2}^{r_B}|v(K_j)|+|h(K)|^2+2|h(K)|n_B-2|h(K)|^2r_B \qquad \text{(i)}$$

$$-2|h(K)|\sum_{j=2}^{r_B}|v(K_j)|-|h(K)|-n_B+r_B|h(K)|+\sum_{j=2}^{r_B}|v(K_j)|$$

As there are no *v(K₁)* terms (i) even after expansion then differentiating this with respect to $|v(K_1)|$ gives:

$$\frac{\partial|s(Q_1)|}{\partial|v(K_1)|}=0 \qquad \text{(ii)}$$

For *i > 1*, differentiating this with respect to $|v(K_i)|$ gives:

$$\frac{\partial|s(Q_1)|}{\partial|v(K_{i,i>2})|}=2\sum_{j=2}^{r_B}|v(K_j)|-2n_B+2|h(K)|r_B-2|h(K)|+1 \qquad \text{(iii)}$$

By proposition 10.7, the number of violational elements in the first disjoint primary set, $|v(K_1)|$, is given by:

$$|v(K_1)|=n_B-\sum_{j=2}^{r_B}|v(K_j)|-|h(K)|r_B \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives:

$$\frac{\partial|s(Q_1)|}{\partial|v(K_{i,i>2})|}=-2|v(K_1)|-2|h(K)|+1 \qquad \text{(v)}$$

Taking the sum of the potential coupling of Q_1 with respect to the number of violational elements in all disjoint primary sets of *B* gives:

$$\sum_{i=1}^{r_B} \frac{\partial|s(Q_1)|}{\partial|v(K_i)|} = \frac{\partial|s(Q_1)|}{\partial|v(K_1)|} + \sum_{i=2}^{r_B} \frac{\partial|s(Q_1)|}{\partial|v(K_{i,i>2})|} \qquad \text{(vi)}$$

Substituting (ii) and (v) into (vi) gives:

$$\sum_{i=1}^{r_B} \frac{\partial|s(Q_1)|}{\partial|v(K_i)|} = 0 + \sum_{i=2}^{r_B} (-2|v(K_1)| - 2|h(K)| + 1)$$

$$= \sum_{i=2}^{r_B} -2|v(K_1)| - \sum_{i=2}^{r_B} 2|h(K)| + \sum_{i=2}^{r_B} 1$$

$$= -2|v(K_1)|(r_B - 1) - 2|h(K)|(r_B - 1) + r_B - 1$$

$$= -2r_B|v(K_1)| + 2|v(K_1)| - 2|h(K)|r_B + 2|h(K)| + r_B - 1$$

QED

Proposition 10.12

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, differentiating the potential coupling of the m^{th} primary set, $|s(Q_m)|$, where $m > 1$, with respect the number of violational elements in any randomly chosen i^{th} disjoint primary gives:

$$\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_i)|} = \begin{Bmatrix} i<m : 0 \\ i=m : |h(K)| - 1 + \sum_{j=1}^{m} |v(K_j)| \\ i>m : -|v(K_m)| - |h(K)| \end{Bmatrix}$$

Proof:

By proposition 10.6, the potential coupling of the m^{th} primary set, $|s(Q_m)|$, is given by:

$$|s(Q_m)| = |h(K)|^2 + |h(K)||v(K_m)| - |h(K)| - |v(K_m)| + n_B|v(K_m)| - |h(K)|r_B|v(K_m)| - |v(K_m)| \sum_{j=m+1}^{r_B} |v(K_j)|$$
$$+ |h(K)|n_B - |h(K)|^2 r_B - |h(K)| \sum_{j=m+1}^{r_B} |v(K_j)|$$

(i)

We now wish to differentiate (i) with $m > 1$ with respect to the number of violational elements in any randomly chosen i^{th} disjoint primary set, but there is a problem: the solution is not unique, and instead depends on how *i* relates to *m*. Thus we must consider three separate solutions.

Firstly, differentiating (i) with respect to the number of violational elements in the i^{th} disjoint primary set such that $i < m$ gives:

$$\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_{i,i<m})|} = 0 \qquad \text{(i)}$$

Differentiating (i) with respect to the number of violational elements in the i^{th} disjoint primary set such that $i = m$ gives:

$$\frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i=m})|} = |h(K)| - 1 + n_B - |h(K)| r_B - \sum_{j=m+1}^{r_B} |v(K_j)| \qquad \text{(ii)}$$

By proposition 10.3, the following holds:

$$\sum_{j=1}^{m-1} |v(K_j)| = n_B - |h(K)| r_B - \sum_{j=m}^{r_B} |v(K_j)| \qquad \text{(iii)}$$

And from (iii) it follows that:

$$\sum_{j=1}^{m} |v(K_j)| = n_B - |h(K)| r_B - \sum_{j=m+1}^{r_B} |v(K_j)| \qquad \text{(iv)}$$

Substituting (iv) into (ii) gives:

$$\frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i=m})|} = |h(K)| - 1 + \sum_{j=1}^{m} |v(K_j)| \qquad \text{(v)}$$

Finally, differentiating (i) with respect to the number of violational elements in the i^{th} disjoint primary set such that $i > m$ gives:

$$\frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i>m})|} = -|v(K_m)| - |h(K)| \qquad \text{(vi)}$$

QED

Proposition 10.13

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set such that $i = m$ gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i=m})|} = |h(K)| r_B - |h(K)| - r_B + 1 - |v(K_1)| + n_B r_B - |h(K)| r_B^2 - \sum_{j=1}^{r_B-1} j |v(K_{j+1})|$$

Proof:

By equation (v) of proposition 10.12, differentiating the potential coupling of the m^{th} primary set, $|s(Q_m)|$, where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set such that $i = m$ gives:

$$\frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i=m})|} = |h(K)| - 1 + \sum_{j=1}^{m} |v(K_j)| \qquad \text{(i)}$$

As we are only interested in the case of , then taking the sum over all K_i will just result in those terms where $i = m$; in other words, there is only one i^{th} term per *m* in which we are interested, or:

$$\sum_{i=2}^{r_B} \frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i=m})|} = \frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_{i,i=m})|} \qquad \text{(ii)}$$

Taking the sum from $m=2..r_B$ of (ii), however, gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i=m})|} = \sum_{m=2}^{r_B} (|h(K)| - 1 + \sum_{j=1}^{m} |v(K_j)|)$$

$$= \sum_{m=2}^{r_B} |h(K)| - \sum_{m=2}^{r_B} 1 + \sum_{m=2}^{r_B} \sum_{j=1}^{m} |v(K_j)|$$

$$=$$

$$|h(K)|(r_B-1)-(r_B-1)+(r_B-1)|v(K_1)|+(r_B-1)|v(K_2)|+(r_B-2)|v(K_3)|+(r_B-3)|v(K_4)|+...+|v(K_{r_B})|$$

$$= \ |h(K)|r_B - |h(K)| - r_B + 1 + (r_B-1)|v(K_1)| + \sum_{j=1}^{r_B-1} (r_B - j)|v(K_{j+1})|$$

$$= \ |h(K)|r_B - |h(K)| - r_B + 1 + r_B|v(K_1)| - |v(K_1)| + r_B \sum_{j=1}^{r_B-1} |v(K_{j+1})| - \sum_{j=1}^{r_B-1} j|v(K_{j+1})| \qquad \text{(iii)}$$

But:

$$\sum_{j=1}^{r_B-1} |v(K_{j+1})| = \sum_{j=2}^{r_B} |v(K_j)|$$

And:

$$r_B \sum_{j=2}^{r_B} |v(K_j)| + r_B|v(K_1)| = r_B \sum_{j=1}^{r_B} |v(K_j)| \qquad \text{(iv)}$$

Substituting (iv) into (iii) gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i=m})|} = |h(K)|r_B - |h(K)| - r_B + 1 - |v(K_1)| + r_B \sum_{j=1}^{r_B} |v(K_j)| - \sum_{j=1}^{r_B-1} j|v(K_{j+1})| \qquad \text{(v)}$$

By equation (iii) of proposition 10.7:

$$n_B = \sum_{j=1}^{r_B} |v(K_j)| + |h(K)|r_B$$

$$\sum_{j=1}^{r_B} |v(K_j)| = n_B - |h(K)|r_B \qquad \text{(vi)}$$

Substituting (vi) into (v) gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i=m})|} = |h(K)|r_B - |h(K)| - r_B + 1 - |v(K_1)| + r_B(n_B - |h(K)|r_B) - \sum_{j=1}^{r_B-1} j|v(K_{j+1})|$$

$$= \ |h(K)|r_B - |h(K)| - r_B + 1 - |v(K_1)| + n_B r_B - |h(K)|r_B^2 - \sum_{j=1}^{r_B-1} j|v(K_{j+1})|$$

QED

Proposition 10.14

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set such that *i > m* gives:

$$\sum_{m=2}^{r_B-1}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_{i,i>m})|}=-n_B r_B+r_B|v(K_1)|+r_B|v(K_{r_B})|+2|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|$$

Proof:

By equation (vi) of proposition 10.12, differentiating the potential coupling of the m^{th} primary set, $|s(Q_m)|$, where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set such that *i > m* gives:

$$\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_{i,i>m})|}=-|v(K_m)|-|h(K)| \qquad \text{(i)}$$

We are interested in evaluating (i) over all *i, m > 1*. We note, however, that *i > m* only for values of *m* that are less than r_B, so the upper value of *m* is r_B - *1*. That is, we need to calculate:

$$\sum_{m=2}^{r_B-1}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_{i,i>m})|}$$

Let us first take the sum of (i) over all *i > m,* giving:

$$\sum_{i=2}^{r_B}\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_{i,i>m})|}=\sum_{i=2}^{r_B}(-|v(K_m)|-|h(K)|) \qquad \text{(ii)}$$

As the neither of the terms on the right-hand side of (i) is a function of *i* and there are *(r_B - m)* terms in which *i > m*, then:

$$\sum_{i=2}^{r_B}\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_{i,i>m})|}=(-|v(K_m)|-|h(K)|)(r_B-m)$$

$$= \quad -r_B|v(K_m)|-|h(K)|r_B+m|v(K_m)|+m|h(K)| \qquad \text{(iii)}$$

Taking the sum from $m=2..r_B-1$ of (iii) gives:

$$\sum_{m=2}^{r_B-1}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_{i,i>m})|}=\sum_{m=2}^{r_B}(-r_B|v(K_m)|-|h(K)|r_B+m|v(K_m)|+m|h(K)|)$$

$$= \quad -\sum_{m=2}^{r_B-1}r_B|v(K_m)|-\sum_{m=2}^{r_B-1}|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|$$

$$= \quad -r_B\sum_{m=2}^{r_B-1}|v(K_m)|-|h(K)|r_B(r_B-2)+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)| \qquad \text{(iv)}$$

By equation (iii) of proposition 10.7:

$$n_B = \sum_{j=1}^{r_B} |v(K_j)| + |h(K)| r_B$$

$$\sum_{j=1}^{r_B} |v(K_j)| = n_B - |h(K)| r_B$$

$$\sum_{j=2}^{r_B} |v(K_j)| + |v(K_1)| = n_B - |h(K)| r_B$$

$$\sum_{j=2}^{r_B-1} |v(K_j)| + |v(K_1)| + |v(K_{r_B})| = n_B - |h(K)| r_B$$

$$\sum_{j=2}^{r_B-1} |v(K_j)| = n_B - |h(K)| r_B - |v(K_1)| - |v(K_{r_B})| \qquad \text{(v)}$$

Substituting (v) into (iv) gives:

$$\sum_{m=2}^{r_B-1} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i>m})|} = -r_B (n_B - |h(K)| r_B - |v(K_1)| - |v(K_{r_B})|) - |h(K)| r_B (r_B - 2) + \sum_{m=2}^{r_B-1} m |v(K_m)| + \sum_{m=2}^{r_B-1} m |h(K)|$$

$$=$$

$$-n_B r_B + |h(K)| r_B^2 + r_B |v(K_1)| + r_B |v(K_{r_B})| - |h(K)| r_B^2 + 2 |h(K)| r_B + \sum_{m=2}^{r_B-1} m |v(K_m)| + \sum_{m=2}^{r_B-1} m |h(K)|$$

$$= \quad -n_B r_B + r_B |v(K_1)| + r_B |v(K_{r_B})| + 2 |h(K)| r_B + \sum_{m=2}^{r_B-1} m |v(K_m)| + \sum_{m=2}^{r_B-1} m |h(K)|$$

QED

Proposition 10.15

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set where *i > 1* gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_i)|} = -|h(K)| - r_B + 1 - 2 |v(K_1)| - |h(K)| r_B^2 + n_B + r_B |v(K_1)| + 2 |h(K)| r_B + \sum_{m=2}^{r_B-1} m |h(K)|$$

Proof:

By definition:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_i)|} = \sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i<m})|} + \sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i=m})|} + \sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial |s(Q_m)|}{\partial |v(K_{i,i>m})|} \qquad \text{(i)}$$

By proposition 10.12:

$$\frac{\partial |s(Q_{m,m>1})|}{\partial |v(K_i)|} = \{ i < m : 0 \} \qquad \text{(ii)}$$

By proposition 10.13, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where $m > 1$, with respect to the number of violational elements in the i^{th} disjoint primary set such that $i = m$ gives:

$$\sum_{m=2}^{r_B}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_{i,i=m})|}=|h(K)|r_B-|h(K)|-r_B+1-|v(K_1)|+n_B r_B-|h(K)|r_B^2-\sum_{j=1}^{r_B-1}j|v(K_{j+1})|$$

(iii)

By proposition 10.14, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where $m > 1$, with respect to the number of violational elements in the i^{th} disjoint primary set such that $i > m$ gives:

$$\sum_{m=2}^{r_B-1}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_{i,i>m})|}=-n_B r_B+r_B|v(K_1)|+r_B|v(K_{r_B})|+2|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|$$

(iv)

Substituting (ii), (iii) and (iv) into (i) gives:

$$\sum_{m=2}^{r_B}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}=|h(K)|r_B-|h(K)|-r_B+1-|v(K_1)|+n_B r_B-|h(K)|r_B^2-\sum_{j=1}^{r_B-1}j|v(K_{j+1})|$$

$$-n_B r_B+r_B|v(K_1)|+r_B|v(K_{r_B})|+2|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|$$

$$=\begin{array}{c}-|h(K)|-r_B+1-|v(K_1)|+n_B r_B-|h(K)|r_B^2-\sum_{j=1}^{r_B-1}j|v(K_{j+1})|\\ -n_B r_B+r_B|v(K_1)|+r_B|v(K_{r_B})|+3|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|\end{array} \quad \text{(v)}$$

By definition:

$$\sum_{j=1}^{r_B-1}j|v(K_{j+1})|=\sum_{m=1}^{r_B-1}m|v(K_{m+1})|$$

$$\sum_{j=1}^{r_B-1}j|v(K_{j+1})|=\sum_{m=2}^{r_B}(m-1)|v(K_m)|$$

$$=\sum_{m=2}^{r_B}m|v(K_m)|-\sum_{m=2}^{r_B}|v(K_m)|$$

$$\sum_{j=1}^{r_B-1}j|v(K_{j+1})|=\sum_{m=2}^{r_B-1}m|v(K_m)|+r_B|v(K_{r_B})|-\sum_{m=2}^{r_B}|v(K_m)| \quad \text{(vi)}$$

Substituting (vi) into (v) gives:

$$\sum_{m=2}^{r_B}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}=-|h(K)|-r_B+1-|v(K_1)|+n_B r_B-|h(K)|r_B^2$$

$$-\sum_{m=2}^{r_B-1}m|v(K_m)|-r_B|v(K_{r_B})|+\sum_{m=2}^{r_B}|v(K_m)|$$

$$-n_B r_B+r_B|v(K_1)|+r_B|v(K_{r_B})|+3|h(K)|r_B+\sum_{m=2}^{r_B-1}m|v(K_m)|+\sum_{m=2}^{r_B-1}m|h(K)|$$

$$= -|h(K)| - r_B + 1 - |v(K_1)| - |h(K)|r_B^2 + \sum_{m=2}^{r_B} |v(K_m)| + r_B|v(K_1)| + 3|h(K)|r_B + \sum_{m=2}^{r_B-1} m|h(K)|$$

(vii)

By equation (iii) of proposition 10.7:

$$n_B = \sum_{m=1}^{r_B} |v(K_m)| + |h(K)|r_B$$

$$\sum_{m=1}^{r_B} |v(K_m)| = n_B - |h(K)|r_B$$

$$\sum_{m=2}^{r_B} |v(K_m)| + |v(K_1)| = n_B - |h(K)|r_B$$

$$\sum_{m=2}^{r_B} |v(K_m)| = n_B - |h(K)|r_B - |v(K_1)| \qquad \text{(viii)}$$

Substituting (viii) into (vii) gives:

$$\sum_{m=2}^{r_B} \sum_{i=2}^{r_B} \frac{\partial|s(Q_m)|}{\partial|v(K_i)|} = -|h(K)| - r_B + 1 - |v(K_1)| - |h(K)|r_B^2 + n_B$$

$$-|h(K)|r_B - |v(K_1)| + r_B|v(K_1)| + 3|h(K)|r_B + \sum_{m=2}^{r_B-1} m|h(K)|$$

$$= -|h(K)| - r_B + 1 - 2|v(K_1)| - |h(K)|r_B^2 + n_B + r_B|v(K_1)| + 2|h(K)|r_B + \sum_{m=2}^{r_B-1} m|h(K)|$$

QED

Proposition 10.16

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the number of violational elements in the first disjoint primary set that minimises the branch's potential coupling is given by:

$$|v(K_1)| = \frac{n_B}{r_B} - \frac{(r_B+1)|h(K)|}{2}$$

Proof:

By proposition 10.10, the potential coupling of *B* is minimised when:

$$\sum_{i=1}^{r_B} \frac{\partial|s(Q_1)|}{\partial|v(K_i)|} + \sum_{k=2}^{r_B} \sum_{i=1}^{r_B} \frac{\partial|s(Q_k)|}{\partial|v(K_i)|} = 0 \qquad \text{(i)}$$

By proposition 10.15, the sum of the potential coupling of all primary sets $|s(Q_m)|$ where *m > 1*, with respect to the number of violational elements in the i^{th} disjoint primary set where *i > 1* gives:

$$\sum_{m=2}^{r_B}\sum_{i=2}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}=-|h(K)|-r_B+1-2|v(K_1)|-|h(K)|r_B^2+n_B+r_B|v(K_1)|+2|h(K)|r_B+\sum_{m=2}^{r_B-1}m|h(K)|$$

(ii)

By proposition 10.11, the sum of the potential coupling of the first primary set Q_1 with respect to the number of violational elements in all disjoint primary sets of B is given by:

$$\sum_{i=1}^{r_B}\frac{\partial|s(Q_1)|}{\partial|v(K_i)|}=-2r_B|v(K_1)|+2|v(K_1)|-2|h(K)|r_B+2|h(K)|+r_B-1 \quad \text{(iii)}$$

Substituting (ii) and (iii) into (i) gives:

$$-|h(K)|-r_B+1-2|v(K_1)|-|h(K)|r_B^2+n_B+r_B|v(K_1)|+2|h(K)|r_B+\sum_{m=2}^{r_B-1}m|h(K)|$$
$$-2r_B|v(K_1)|+2|v(K_1)|-2|h(K)|r_B+2|h(K)|+r_B-1=0$$
$$= \quad -|h(K)|r_B^2+n_B+|h(K)|\sum_{m=2}^{r_B-1}m-r_B|v(K_1)|+|h(K)| \quad \text{(iv)}$$

By definition:

$$\sum_{m=2}^{r_B-1}m=\sum_{m=1}^{r_B}m-1-r_B$$
$$= \quad \frac{r_B(r_B+1)}{2}-1-r_B \quad \text{(v)}$$

Substituting (v) into (iv) gives:

$$-|h(K)|r_B^2+n_B+|h(K)|(\frac{r_B(r_B+1)}{2}-1-r_B)-r_B|v(K_1)|+|h(K)|=0$$
$$= \quad -|h(K)|r_B^2+n_B+\frac{|h(K)|r_B(r_B+1)}{2}-|h(K)|-|h(K)|r_B-r_B|v(K_1)|+|h(K)|$$
$$= \quad -2|h(K)|r_B^2+2n_B+|h(K)|r_B(r_B+1)-2|h(K)|-2|h(K)|r_B-2r_B|v(K_1)|+2|h(K)|$$
$$= \quad -2|h(K)|r_B^2+2n_B+|h(K)|r_B^2+|h(K)|r_B-2|h(K)|-2|h(K)|r_B-2r_B|v(K_1)|+2|h(K)|$$
$$= \quad -|h(K)|r_B^2+2n_B-|h(K)|r_B-2r_B|v(K_1)|$$
$$2r_B|v(K_1)|=2n_B-|h(K)|r_B^2-|h(K)|r_B$$
$$|v(K_1)|=\frac{n_B}{r_B}-\frac{|h(K)|r_B+|h(K)|}{2}$$
$$= \quad \frac{n_B}{r_B}-\frac{(r_B+1)|h(K)|}{2}$$

QED

Proposition 10.17

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the number of violational elements in the i^{th} disjoint primary set that minimises the branch's potential coupling is given by:

$$|v(K_i)|=|v(K_1)|+|h(K)|i-|h(K)|$$

Proof:

By definition I SHOULD PROVE THIS ????????????????????????xxx, the potential coupling of branch B is equal to the sum of the potential coupling of all its primary sets, or:

$$|s(B)|=\sum_{m=1}^{r_B}|s(Q_m)| \qquad \text{(i)}$$

Choosing an arbitrary *i* such that $1<i\leq r_B$ we can re-write (i) as:

$$|s(B)|=|s(Q_1)|+\sum_{m=2}^{i-1}|s(Q_m)|+|s(Q_i)|+\sum_{m=i+1}^{r_B}|s(Q_m)| \qquad \text{(ii)}$$

Taking the derivative of both sides of (ii) with respect to $|v(K_i)|$ gives:

$$\frac{\partial|s(B)|}{\partial|v(K_i)|}=\frac{\partial|s(Q_1)|}{\partial|v(K_i)|}+\sum_{m=2}^{i-1}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|}+\frac{\partial|s(Q_i)|}{\partial|v(K_i)|}+\sum_{m=i+1}^{r_B}\frac{\partial|s(Q_m)|}{\partial|v(K_i)|} \qquad \text{(iii)}$$

Setting (iii) equal to *0* will yield the minimum potential coupling of branch B with respect to the number of violational elements in the K_i.

By proposition 10.12, differentiating the potential coupling of the m^{th} primary set, $|s(Q_m)|$, where *m* > *1*, with respect the number of violational elements in any randomly chosen i^{th} disjoint primary gives:

$$\frac{\partial|s(Q_{m,m>1})|}{\partial|v(K_i)|}=\begin{Bmatrix} i<m:0 \\ i=m:|h(K)|-1+\sum_{j=1}^{i}|v(K_j)| \\ i>m:-|v(K_m)|-|h(K)| \end{Bmatrix} \qquad \text{(iv)}$$

By equation (v) in proposition 10.11,

$$\frac{\partial|s(Q_1)|}{\partial|v(K_{i,i>2})|}=-2|v(K_1)|-2|h(K)|+1 \qquad \text{(v)}$$

Substituting (iv) and (v) into (iii) gives:

$$\frac{\partial|s(B)|}{\partial|v(K_i)|}=-2|v(K_1)|-2|h(K)|+1+\sum_{m=2}^{i-1}(-|v(K_m)|-|h(K)|)+|h(K)|-1+\sum_{j=1}^{i}|v(K_j)|+0=0$$

$$= \; -2|v(K_1)|-2|h(K)|+1-\sum_{m=2}^{i-1}|v(K_m)|-\sum_{m=2}^{i-1}|h(K)|+|h(K)|-1+\sum_{j=1}^{i}|v(K_j)|$$

$$=$$

$$-2|v(K_1)|-2|h(K)|+1-\sum_{m=1}^{i}|v(K_m)|+|v(K_1)|+|v(K_i)|-|h(K)|(i-2)+|h(K)|-1+\sum_{j=1}^{i}|v(K_j)|$$

$$= -2|v(K_1)|-2|h(K)|+1+|v(K_1)|+|v(K_i)|-|h(K)|i+2|h(K)|+|h(K)|-1$$

$$= -|v(K_1)|+|v(K_i)|-|h(K)|i+|h(K)|$$

$$|v(K_i)|=|v(K_1)|+|h(K)|i-|h(K)|$$

QED

Proposition 10.18

Given a branch *B* of n_B elements and of r_B disjoint primary sets, with each disjoint primary set having an information-hiding of $|h(K)|$, the number of violational elements in the i^{th} disjoint primary set that minimises the branch's potential coupling is given by:

$$|v(K_i)|=\frac{n_B}{r_B}+\frac{(2i-3-r_B)|h(K)|}{2}$$

Proof:

By proposition 10.16, the number of violational elements in the first disjoint primary set that minimises the branch's potential coupling is given by:

$$|v(K_1)|=\frac{n_B}{r_B}-\frac{(r_B+1)|h(K)|}{2} \quad \text{(i)}$$

By proposition 10.17, the number of violational elements in the i^{th} disjoint primary set that minimises the branch's potential coupling is given by:

$$|v(K_i)|=|v(K_1)|+|h(K)|i-|h(K)| \quad \text{(ii)}$$

Substituting (i) into into (ii) gives:

$$|v(K_i)|=\frac{n_B}{r_B}-\frac{(r_B+1)|h(K)|}{2}+|h(K)|i-|h(K)|$$

$$= \frac{n_B}{r_B}+\frac{-|h(K)r_B|-|h(K)|+2|h(K)|i-2|h(K)|}{2}$$

$$= \frac{n_B}{r_B}+\frac{2|h(K)|i-3|h(K)|-|h(K)r_B|}{2}$$

$$= \frac{n_B}{r_B}+\frac{(2i-3-r_B)|h(K)|}{2}$$

QED

6. References

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